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The longitudinal drift velocity of a sheared dilute suspension of spheres

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Abstract

We present the calculation of the particle drift velocity for a dilute, neutrally buoyant suspension of spheres under the action of shear when inertia and Brownian effects are negligible, resulting from the effect of an imposed concentration gradient. Using a renormalization technique employed previously, we found that the particle drift velocity $\langle \mathbf{U} \rangle$ is proportional to the concentration gradient $\nabla\phi$ through the relation $\langle \mathbf{U} \rangle = \beta a^2 \mathbf{E} \cdot \nabla\phi + O(\phi\nabla\phi)$, where a is the radius of the spheres and $\mathbf{E}$ is the fluid rate of strain, while β is an $O(1)$ constant that depends on the angular velocity of the fluid flow. In particular, $\beta = 2.40$ for simple shear flow, and $\beta = 3.12$ for pure straining flow. Finally, combining the expression of the drift velocity with that of the self-diffusivity, we determined the shear-induced particle volumetric flux and cross gradient diffusivity for the case of simple shear flow. © 1999 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Shear-induced gradient diffusion; Suspensions

1. Introduction

In this article we study the shear-induced diffusion of non-Brownian, neutrally buoyant particles suspended in a viscous fluid undergoing shear flow. This phenomenon is responsible, among others, for the viscous resuspension of heavy particles under the influence of shear (Leighton and Acrivos, 1986), which has found important applications in the design of super settlers. The mechanism that is responsible for shear-induced diffusion is well known: particles

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suspended in a viscous fluid under conditions in which the flow is laminar and the particle Reynolds number is vanishingly small tend to move from high collision rate regions to low. When the shear rate is constant, a test particle appears to undergo a random walk as it interacts with its neighboring particles, so that the process can be described through a diffusivity proportion to the shear rate γ and the square of the particle radius a. Now, two different diffusion coefficients can be defined, namely self-diffusivity and gradient diffusivity. The former is defined as the temporal growth rate of the mean square displacement of a randomly chosen tagged particle in a uniform concentration field, while the latter is the ratio between mass flux and concentration gradient. These two diffusivities are generally different from each other whenever the suspended particles interact hydrodynamically. For example, self- and gradient diffusivities of Brownian particles in a quiescent dilute suspension are equal to $D_0(1-1.83\phi)$ and $D_0(1+1.45\phi)$, respectively (Batchelor, 1976), where D_0 is the particle molecular diffusivity and ϕ is the particle volume fraction.

Shear-induced diffusion was first observed experimentally by Eckstein et al. (1977). Later, quantitative measurements were performed by Leighton and Acrivos (1987a,b), who obtained values for the shear-induced self-diffusivity which are in qualitative agreement with the numerical results for monolayer suspensions obtained by Brady and Bossis (1988). Analytically, shear-induced self-diffusion was studied in the dilute limit by Acrivos et al. (1992) and by Wang et al. (1996), who determined the diffusion coefficient in the direction of the fluid velocity and in that perpendicular to it, respectively.

Despite its obvious importance in practical applications, the phenomenon of shear-induced gradient diffusion appears to have been studied only experimentally, cf. Leighton and Acrivos (1987a,b) or Phillips et al. (1992), with the result that no analytical expressions currently exist for the corresponding gradient diffusivity, which have been determined either from fundamental analysis or from ab initio computations, except for that of a monolayer of spheres (Wang et al., 1998).

In this work, we derive an expression for the shear-induced drift velocity of a dilute monodisperse suspension of spheres, that is the mean velocity of a test sphere, due to an imposed particle concentration gradient at infinity. In the case of simple shear flow, this leads us to determine the shear-induced cross gradient diffusivity, which is the ratio between the particle flux *in the direction of the fluid velocity* and an imposed gradient of the particle concentration in a direction perpendicular to the fluid velocity. To be sure, this so-called longitudinal diffusion is not very important from a practical standpoint, as any lateral diffusion would lead to differential convection in the streamwise direction, which seems likely to dominate longitudinal diffusion. However, compared with the calculations of the coefficient of transverse shear-induced gradient diffusion (Wang et al. 1998), the analysis that we present here has the advantage of clarifying the meaning of the drift particle velocity. This is due to the fact that this calculation is much simpler than that of the transverse diffusivity, involving only pairwise interactions among the suspended particles, and consequently, as the problem can be solved analytically, the physical meaning of each term in the expression of the drift velocity can be clearly pointed out. In addition, as we learned in the study of the self-diffusivity, which we considered earlier (Acrivos et al., 1992; Wang et al., 1996), solving the longitudinal case can provide us with the theoretical framework to perform subsequent more important numerical calculations.

2. The basic approach

Consider a dilute, monodisperse suspension of force-free and couple-free spherical particles immersed in a fluid with a concentration gradient along the $\hat{\mathbf{e}}_2$-direction. That means that the probability density of finding a particle at location $\mathbf{r} = (x_1, x_2, x_3)$ is equal to $n_0 P(\mathbf{r})$, with:

$$P(\mathbf{r}) = 1 + \frac{1}{\phi_0}\left\langle \frac{\partial \phi}{\partial x_2.} \right\rangle x_2 \tag{1}$$

Here n is the number density, $\langle \partial\phi/\partial x_2 \rangle$ is the imposed mean gradient in the $\hat{\mathbf{e}}_2$-direction of the particle volume fraction $\phi = \frac{4}{3}\pi a^3 n$, and the subscript '0' refers to the value of n and ϕ at the origin. In addition, we assume that the spheres have a radius a small enough that inertia effects can be neglected, while the fluid is incompressible, has viscosity μ, and undergoes a uniform shear flow with velocity $\mathbf{v}(\mathbf{r}) = \gamma x_2 \hat{\mathbf{e}}_1$ along the $\hat{\mathbf{e}}_1$-direction. Clearly, in writing Eq. (1) we have supposed the particle concentration varies over distances which are large relative to a, or, conversely, that $x_2 l = \phi_0 \langle \partial\phi/\partial x_2 \rangle^{-1}$.

The goal of this calculation is to find the mean volumetric flux $\mathbf{J}$ of the suspended particles in terms of the concentration gradient. In turn, the value of $\mathbf{J}$ at the origin depends on the instantaneous mean velocity $\langle \mathbf{U} \rangle$ of a test sphere at the origin, which is given by:

$$\langle \mathbf{U} \rangle = n_0 \int \mathbf{U}(\mathbf{0} \mid \mathbf{r}) P(\mathbf{r} \mid \mathbf{0})\, \mathrm{d}^3\mathbf{r} + O(n_0^2). \tag{2}$$

Here $\mathbf{U}(\mathbf{0}|\mathbf{r})$ is the instantaneous velocity of the test sphere at the origin in the presence of a second sphere at position $\mathbf{r}$, and $P(\mathbf{r}|\mathbf{0})$ denotes the normalized conditional probability of finding the second sphere at $\mathbf{r}$, provided that the test sphere is located at the origin.

The velocity $\mathbf{U}(\mathbf{0}|\mathbf{r})$ may be written as:

$$U_i(\mathbf{0} \mid \mathbf{r}) = \frac{1}{2} E_{jk} x_k \left[A(r) \frac{x_i x_j}{r^2} + B(r) \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \right] \tag{3}$$

where $r = |\mathbf{r}|$, $E_{jk} = \frac{1}{2}\gamma(\delta_{j1}\delta_{k2} + \delta_{j2}\delta_{k1})$ is the uniform rate of strain tensor, while $A(r)$ and $B(r)$ are scalar functions of r, decaying as r^{-3} and r^{-5} as $r \to \infty$, respectively (Batchelor and Green, 1972a).

Clearly, Eqs. (2) and (3) show that the mean velocity of the test sphere in a uniformly distributed suspension, i.e. with $P(\mathbf{r}|\mathbf{0}) = H(r - 2a)$, where H is the Heaviside function, is identically zero, so that $\langle \mathbf{U} \rangle$ is determined only by the deviation of $P(\mathbf{r}|\mathbf{0})$ from a constant value. Now, the conditional probability $P(\mathbf{r}|\mathbf{0})$ is not known a priori, as it is the solution of a two-particle convection problem, and can be determined following the method of Batchelor and Green (1972b). However, we prefer to defer this calculation to the next section, and here focus on describing the general method to solve the integral in Eq. (2). Accordingly, we assume that $P(\mathbf{r}|\mathbf{0})$ is a known quantity, referred to as the 'unperturbed' conditional probability, $P_\infty(\mathbf{r}|\mathbf{0})$, which is given by:

$$p_\infty(\mathbf{r} \mid \mathbf{0}) = \left[1 + \frac{1}{\phi_0}\left\langle \frac{\partial \phi}{\partial x_2} \right\rangle x_2 \right] H(r - 2a). \tag{4}$$

Substituting Eqs. (4) and (3) into Eq. (2) we obtain:

$$\langle U_2\rangle_\infty = \langle U_3\rangle_\infty = O(n_0^2) \tag{5}$$

and

$$\langle U_1\rangle_\infty = \alpha\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{6}$$

with

$$\alpha = \frac{3}{16\pi}\int_{\hat{r}\geq 2}\left\{2[A(\hat{r})-B(\hat{r})]\left(\frac{\hat{x}_1\hat{x}_2}{\hat{r}}\right)^2 + B(\hat{r})\hat{x}_2^2\right\}\,\mathrm{d}^3\hat{\mathbf{r}} \tag{7}$$

where $\hat{\mathbf{r}} = \mathbf{r}/a$, and we have taken it for granted that the constant term in $P_\infty(\mathbf{r}|\mathbf{0})$ does not contribute to $\langle U_1\rangle$. The subscript '∞' in Eqs. (5) and (6) for the mean velocities refers to the fact that these quantities are determined via the unperturbed, and not the 'exact', conditional probability distribution. Eq. (5) indicates that, as expected, the transverse drift velocities are $O(\phi_0^2)$ quantities, and can be determined only if three-particle interactions are taken into account. However, since, we are only interested in the $O(\phi_0)$-terms of the volumetric flux, we shall concentrate on the determination of $\langle \mathrm{U}_1\rangle$. The main difficulty which must be overcome, however, is that the integral in Eq. (7) diverges as $\int\hat{r}\,\mathrm{d}\hat{r}$, hence we need to resolve the singularity through renormalization.

We apply the renormalization procedure developed by Batchelor (1972) to determine $\langle U_1\rangle$ in Eq. (2). First we follow Batchelor (1972) and by applying Faxen's law let

$$\mathbf{U}(\mathbf{0}\mid\mathbf{r}) = \mathbf{u}(\mathbf{0}\mid\mathbf{r}) + \left(\frac{a^2}{6}\right)\nabla^2\mathbf{u}(\mathbf{0}\mid\mathbf{r}) + \mathbf{W}(\mathbf{0}\mid\mathbf{r}) \tag{8}$$

where $\mathbf{u}(\mathbf{0}|\mathbf{r})$ is the fluid velocity at the origin in the presence of a particle at position $\mathbf{r}$, and $\mathbf{W}(\mathbf{0}|\mathbf{r})$ is a remainder. Then, considering that the constant term, 1, can be subtracted out from $P_\infty(\hat{\mathbf{r}}|\mathbf{0})$ (recall that the mean velocity of the test sphere in a uniform suspension is identically zero), we see on substituting Eq. (8) into Eq. (2) that $\langle U_1\rangle_\infty$ equals the sum of three contributions, i.e. $\langle U_1\rangle_\infty = I_{(1)} + I_{(2)} + I_{(3)}$, with

$$I_{(1)} = n_0 a^3\int_{\hat{r}>2} W_1(\mathbf{0}\mid\hat{\mathbf{r}})[P_\infty(\hat{\mathbf{r}}\mid\mathbf{0}) - 1]\,\mathrm{d}^3\hat{\mathbf{r}} \tag{9}$$

$$I_{(2)} = n_0 a^3\int_{\hat{r}>2} u_1(\mathbf{0}\mid\hat{\mathbf{r}})[P_\infty(\hat{\mathbf{r}}\mid\mathbf{0}) - 1]\,\mathrm{d}^3\hat{\mathbf{r}} \tag{10}$$

$$I_{(3)} = n_0\frac{a^5}{6\mu}\int_{\hat{r}>2}\frac{\partial p}{\partial x_1}(\mathbf{0}\mid\hat{\mathbf{r}})[P_\infty(\hat{\mathbf{r}}\mid\mathbf{0}) - 1]\,\mathrm{d}^3\hat{\mathbf{r}} \tag{11}$$

where in Eq. (11) p is the fluid pressure, and use has been made of the Stokes equation $\nabla p = \mu\nabla^2\mathbf{u}$.

The integral in Eq. (9) is converging, since $W_1(\mathbf{0}|\hat{\mathbf{r}})$ decays like $\hat{r}^{-5}$ as $r \to \infty$. In fact,

$$W_1(\mathbf{0} \mid \hat{\mathbf{r}}) = \frac{1}{2}\gamma a[A'(\hat{r}) - B'(\hat{r})]\frac{\hat{x}_1^2\hat{x}_2}{\hat{r}^2} + \frac{1}{4}\gamma a\hat{x}_2 B'(\hat{r}) \tag{12}$$

where the functions $A'(\hat{r})$ and $B'(\hat{r})$ include only those terms of $A(\hat{r})$ and $B(\hat{r})$ which decay like $\hat{r}^{-6}$ or faster. Therefore, on substituting Eqs. (12) and (4) into Eq. (9) we obtain:

$$I_{(1)} = \frac{1}{20}\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle\int_{\hat{r}>2}[2A'(\hat{r}) + 3B'(\hat{r})]\hat{r}^4\ \mathrm{d}\hat{r} \tag{13}$$

which, after numerical integration gives:

$$I_{(1)} = 1.317\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{14}$$

The other integrals in Eqs. (10) and (11) are non-absolutely convergent, and can be renormalized, as was done by Batchelor (1972) for the sedimentation problem, by imposing the constraints that the ensemble average velocity and pressure gradient at the origin both be zero. For a dilute suspension these conditions are equivalent to requiring that

$$\int \mathbf{u}(\mathbf{0} \mid \hat{\mathbf{r}})[P(\hat{\mathbf{r}}) - 1]\ \mathrm{d}^3\hat{\mathbf{r}} = \mathbf{0} \tag{15}$$

and

$$\int \nabla p(\mathbf{0} \mid \hat{\mathbf{r}})[P(\hat{\mathbf{r}}) - 1]\ \mathrm{d}^3\hat{\mathbf{r}} = \mathbf{0} \tag{16}$$

where $\mathbf{u}(\mathbf{0}|\hat{\mathbf{r}})$ and $p(\mathbf{0}|\hat{\mathbf{r}})$ are the velocity and pressure at the origin given at a sphere is located at $\hat{\mathbf{r}}$. Note that the integrals in Eqs. (15) and (16) are evaluated over the whole space, including the region $0 \leqslant \hat{r} \leqslant 2$. Now, subtracting Eq. (15) from Eq. (10) and Eq. (16) from Eq. (11), and noting that $P(\hat{\mathbf{r}}) = P_\infty(\hat{\mathbf{r}}|\mathbf{0})$ for $\hat{r} > 2$, [cf. Eqs. (1) and (4)], we obtain:

$$I_{(2)} = -n_0 a^3\int_{\hat{r}<2} u_1(\mathbf{0} \mid \hat{\mathbf{r}})[P(\hat{\mathbf{r}}) - 1]\ \mathrm{d}^3\hat{\mathbf{r}} \tag{17}$$

and

$$I_{(3)} = -n_0\frac{a^5}{6\mu}\int_{\hat{r}<2}\frac{\partial p}{\partial x_1}(\mathbf{0} \mid \hat{\mathbf{r}})[P(\hat{\mathbf{r}}) - 1]\ \mathrm{d}^3\hat{\mathbf{r}} \tag{18}$$

First, we calculate $I_{(2)}$ by substituting Eq. (1) for $P(\mathbf{r})$ into Eq. (17) and decomposing the resulting integral as the sum

$$I_{(2)} = -I'_{(2)} - I''_{(2)} = -\frac{3a}{4\pi}\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle\left(\int_{\hat{r}<1} + \int_{1<\hat{r}<2}\right)u_1(\mathbf{0} \mid \hat{\mathbf{r}})\hat{x}_2\ \mathrm{d}^3\hat{\mathbf{r}} \tag{19}$$

Now, $I'_{(2)}$ can be easily evaluated considering that, when $\hat{r} < 1$, the origin lies within the particle located at $\hat{\mathbf{r}}$, so that, as the particle rotates with angular velocity $\mathbf{\Omega} = -\frac{1}{2}\gamma\hat{\mathbf{e}}_3$, we find: $\mathbf{u}(\hat{\mathbf{r}}) = \gamma a[\hat{x}_2\hat{\mathbf{e}}_1 + \frac{1}{2}\hat{\mathbf{e}}_3 \times \hat{\mathbf{r}}]$. Therefore,

$$I'_{(2)} = \frac{3}{8\pi}\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle\int_{\hat{r}<1}\hat{x}_2^2\;\mathrm{d}^3\mathbf{r} = \frac{1}{10}\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{20}$$

The integral $I''_{(2)}$ in Eq. (19) can also be evaluated easily, since the velocity field at the origin due to the presence of an isolated sphere at $\hat{\mathbf{r}}$ is

$$u_1(\hat{\mathbf{r}}) = \frac{1}{2}\gamma a[5\hat{x}_1^2\hat{x}_2(\hat{r}^{-5} - \hat{r}^{-7}) + \hat{x}_2\hat{r}^{-5}] \tag{21}$$

Consequently, on performing the integration (19) we find $I''_{(2)} = \frac{3}{4}\gamma a^2\langle\partial\phi/\partial x_2\rangle$, and summing $I'_{(2)}$ and $I''_{(2)}$ we obtain:

$$I_{(2)} = -\frac{17}{20}\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{22}$$

Finally, we turn to evaluating $I_{(3)}$. Substituting Eq. (1) into Eq. (18) and applying the divergence theorem we obtain:

$$I_{(3)} = -\frac{a^2}{8\pi\mu}\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle\int_{\hat{r}=2}\hat{x}_1\hat{x}_2\hat{r}^{-1}p(\hat{\mathbf{r}})\;\mathrm{d}^2\hat{\mathbf{r}} \tag{23}$$

where $p(\hat{\mathbf{r}}) = -5\mu\gamma\hat{x}_1\hat{x}_2\hat{r}^{-5}$ is the fluid pressure at the origin due to the presence of an isolated sphere at location $\hat{\mathbf{r}}$. Performing the integration (23) we find:

$$I_{(3)} = \frac{1}{6}\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{24}$$

The same result would be obtained had we replaced ∇p with $\mu\nabla^2\mathbf{u}$ in Eq. (16), as in Batchelor (1972). Now, summing Eqs. (14), (22) and (24), we obtain:

$$\langle U_1\rangle_\infty = \alpha_\infty\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle \tag{25}$$

with $\alpha_\infty = 0.634$. In our calculation we have neglected the influence of the hydrodynamic interactions among three or more particles, so that in Eq. (25) all $O(\phi_0^2)$-terms have been neglected. In addition, as the domain of integration was such that $x_2 \ll l = \phi_0\langle\partial\phi/\partial x_2\rangle^{-1}$, we have also neglected terms of $O(a^2\langle\partial\phi/\partial x_2\rangle^2)$.

The expression (25) for the shear-induced drift velocity in the longitudinal direction for simple shear flow can be easily generalized to the case where the mean concentration gradient is a vector $\langle\nabla\phi\rangle$ pointing in any direction, and the unperturbed fluid velocity field is a general shear flow, $v_i(\mathbf{r}) = \Gamma_{ij}x_j$. In this case, the antisymmetric part of the velocity gradient tensor, $\Omega_{ij} = (\Gamma_{ij} - \Gamma_{ji})/2$, corresponds to a rigid body rotation of the suspension and does not contribute to the translation of the test particle at the origin. Therefore, denoting by $E_{ij} = (\Gamma_{ij} + \Gamma_{ji})/2$ the rate of strain tensor, on account of the fact that the test particle velocity is linear in E_{ij}, we can generalize (25) into

$$\langle U_i \rangle_\infty = 2\alpha_\infty a^2 E_{ij} \left\langle \frac{\partial \phi}{\partial x_j} \right\rangle \tag{26}$$

where, as before, terms of $O(\phi^2)$ and $O(|a\nabla\phi|^2)$ have been neglected.

3. The 'exact' calculation

In this section we apply the basic approach described in the previous section to calculate 'exactly' the longitudinal shear-induced drift velocity. To accomplish that, we need first to evaluate the conditional probability function $P(\mathbf{r}|\mathbf{0})$, which in the previous section had been assumed known a priori and equal to its unperturbed expression, $P_\infty(\mathbf{r}|\mathbf{0})$. In fact the presence of the test sphere at the origin will alter the particle distribution in the vicinity of the origin, and so our assumed unperturbed conditional probability cannot be correct.

In the following, $P(\mathbf{r}|\mathbf{0})$ will be found by repeating the analysis of Batchelor and Green (1972b), who determined that quantity in a homogeneous suspension, that is in the absence of any macroscopic concentration gradients, by evaluating the time-dependent conditional probability $p(\mathbf{r},\ t) = P(\mathbf{r}_0 + \mathbf{r},\ t_0 + t|\mathbf{r}_0,\ t_0)$ of finding a particle 1, say, at position $r_0 + \mathbf{r}$ at time $t_0 + t$, provided that another particle 2, say, is at position $\mathbf{r}_0$ at time t_0. This function p, which depends only on the separation vector $\mathbf{r}$ between the two spheres and on the time t, is the solution of the probability conservation equation:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mathbf{V}_p) = 0 \tag{27}$$

with $\mathbf{V}(\mathbf{r})$ being the velocity of particle 2 relative to particle 1, subject to the boundary condition

$$p(\mathbf{r},t) = p_\infty(\mathbf{r}) = 1 + \frac{1}{\phi_0}\mathbf{r} \cdot \langle \nabla \phi \rangle \tag{28}$$

for $r \gg a$. In addition, if at time $t = -\infty$ the separation vector $\mathbf{r}$ between the two particles is infinite, then for any material point in $\mathbf{r}$-space which originates from or ends at infinity we may consider (28) as being our initial condition as well. Now, on noticing that the radial component of $\mathbf{V}$ has the same dependence on the direction of $\mathbf{r}$ as $\nabla \cdot \mathbf{V}$, Batchelor and Green (1972b) rewrote Eq. (27) as:

$$\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{Q}\frac{\mathrm{D}Q}{\mathrm{D}t} \tag{29}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \mathbf{V} \cdot \nabla$ in $\mathbf{r}$-space, while $Q = Q(r)$ is a function of r only, defined through the following equation:

$$\frac{1}{Q}\frac{\mathrm{d}Q}{\mathrm{d}r} = \frac{3(A-B)}{r(1-A)} + \frac{1}{1-A}\frac{\mathrm{d}A}{\mathrm{d}r} \tag{30}$$

with $A(r)$ and $B(r)$ denoting the scalar functions appearing in Eq. (3). To quote Batchelor and

Green (1972b), ‘the meaning of Eq. (29) is that the quantity p/Q is constant for a material point in **r**-space whose velocity is given by **V**’. Hence, since $Q(r)$ is defined within an arbitrary multiplying constant, we can choose this constant to be unity, so that $p=Q$ along a trajectory in **r**-space. Therefore, applying the boundary conditions (28)–(30), we obtain:

$$p(\mathbf{r},t) = P(\mathbf{r} \mid \mathbf{0}) = q(r)P_\infty(\mathbf{r} \mid \mathbf{0}) \tag{31}$$

where $P_\infty(\mathbf{r}|\mathbf{0})$ is defined by Eq. (4), while $q(r)$, with $q(\infty)=1$, is given in Eq. (3.9) of Batchelor and Green (1972b) as

$$q(r) = \frac{1}{1-A} \exp\left\{\int_r^\infty \frac{3(B-A)}{r(1-A)}\,\mathrm{d}r\right\} \tag{32}$$

Consequently, $q(r)$ can be calculated explicitly. One finds that $q(r)-1 \approx \frac{25}{2}\hat{r}^{-6}$ for $\hat{r} \geqslant 3$, $q(r) \approx 0.234(\hat{r}-2)^{-0.781} \log^{-0.29}(\hat{r}-2)^{-1}$ for $\hat{r} \leqslant 2.0025$ (see Fig. 1 in Batchelor and Green, 1972b) while for intermediate values of $\hat{r}$, $q(r)$ has been tabulated in Batchelor and Green (1972b).

In deriving Eqs. (31) and (32) we have assumed that all trajectories originate from infinity, but in the case of simple shear flow some trajectories are closed and occupy a region in space into which open trajectories cannot penetrate. In this region, particle 2 keeps orbiting around the test sphere indefinitely, following a trajectory that is symmetric with respect to the origin (Batchelor and Green, 1972b). Therefore, the mean velocity of the test sphere induced by such periodically orbiting particles is zero, which means that the volume integral in the expression (2) for the mean velocity $\langle \mathbf{U} \rangle$ must be evaluated for values of **r** lying outside the region of closed trajectories. The same conclusion can be reached by noting that the value of the conditional probability $P(\mathbf{r}|\mathbf{0})$ inside the region of closed trajectories can be determined only if the history of the suspension is known before the particles have begun to move. Now, for any such initial particle distribution, after a long time, as the particles keep orbiting the test sphere, the conditional probability will become symmetric with respect to the (x_1, x_3)- and (x_2, x_3)-planes. Therefore, since the region of closed trajectories is also equally symmetric, the contribution of these orbiting particles to Eq. (2) is identically zero.

Now, when the expression (31) is substituted into Eq. (2) and the renormalization procedure described in the previous section is applied, we find again that, as expected, the transverse shear-induced drift velocities are $O(n_0^2)$-quantities,

$$\langle U_2 \rangle = \langle U_3 \rangle = O(n_0^2) \tag{33}$$

while the longitudinal drift velocity is the sum of its ‘unperturbed’ value (25) plus two correction terms,

$$\langle U_1 \rangle = \langle U_1 \rangle_\infty + \langle U_1 \rangle' - \langle U_1 \rangle'' \tag{34}$$

with

$$\langle U_1 \rangle' = n_0 \int_{\hat{r}_{\mathrm{out}}} U_1(\mathbf{0} \mid \hat{\mathbf{r}})[P_\infty(\hat{\mathbf{r}} \mid \mathbf{0}) - 1][q(r) - 1]\,\mathrm{d}^3\hat{\mathbf{r}} \tag{35}$$

$$\langle U_1\rangle'' = n_0\int_{\hat{r}_{\mathrm{in}}} U_1(\mathbf{0}\mid\hat{\mathbf{r}})[P_\infty(\hat{\mathbf{r}}\mid\mathbf{0})-1]\ \mathrm{d}^3\hat{\mathbf{r}} \tag{36}$$

where the integrations are over the volume exterior [Eq. (35)] and interior [Eq. (36)], respectively, to the region of closed trajectories. This region is symmetric with respect to the (x_1, x_3)-plane as well as with respect to the x_2-axis, and is bounded by the surface (Batchelor and Green, 1972b):

$$|\hat{x}_2| = f(\hat{r}) = \left\{\frac{1}{F(\hat{r})}\int_{\hat{r}}^{\infty}\frac{B(\hat{r}')}{1-A(\hat{r}')}F(\hat{r}')\hat{r}'\ \mathrm{d}\hat{r}'\right\}^{\frac{1}{2}} \tag{37}$$

where

$$F(\hat{r}) = \exp\left\{-2\int_{\hat{r}}^{\infty}\frac{A(\xi)-B(\xi)}{1-A(\xi)}\frac{\mathrm{d}\xi}{\xi}\right\} \tag{38}$$

The projection of the surface (37) on the (x_1, x_2)-plane is shown in Fig. 4 of Batchelor and Green (1972a), representing the trajectory of a particle which, far downstream, lies on the x_1-axis. In view of Eq. (37), such a trajectory has the asymptotic form $\hat{x}_2^2 = \frac{16}{9}\hat{r}^{-3}$ for $\hat{r} \gg 1$. Finally, by taking advantage of the axial symmetry of the surface $\hat{x}_2 = f(\hat{r})$, the expression (34) becomes, when the imposed flow is a simple shear,

$$\langle U_1\rangle_\infty = \alpha\gamma a^2\left\langle\frac{\partial\phi}{\partial x_2}\right\rangle, \quad \alpha = (\alpha_\infty + \alpha' - \alpha'') \tag{39}$$

where

$$\alpha' = \frac{3}{4}\int_{\hat{r}_{\min}}^{\infty}\{A(\hat{r})G_{11}(g(\hat{r})) + B(\hat{r})G_{12}(g(\hat{r}))\}[q(\hat{r})-1]\hat{r}^4\ \mathrm{d}\hat{r} \tag{40}$$

$$\alpha'' = \frac{3}{4}\int_{2}^{\infty}\{A(\hat{r})G_{21}(g(\hat{r})) + B(\hat{r})G_{22}(g(\hat{r}))\}\hat{r}^4\ \mathrm{d}\hat{r} \tag{41}$$

where $G_{11}(x) = \frac{1}{3}(1-x^3) - \frac{1}{5}(1-x^5)$, $G_{12}(x) = \frac{1}{5}(1-x^5)$, $G_{21(x)} = \frac{1}{3}x^3 - \frac{1}{5}x^5$ and $G_{22} = \frac{1}{5}x^5$. Here $\hat{r}_{\min} = 2 + 4.155\times10^{-5}$ is the minimum distance from the origin of the surface (37), such that $f(\hat{r}_{\min}) = \hat{r}_{\min}$, while $g(\hat{r}) = f(\hat{r})/\hat{r}$, with $f(\hat{r})$ given by Eq. (37). On computing the integrals in Eqs. (40) and (41), we find $\alpha' = 0.613 \pm 0.005$ and $\alpha'' = 0.045 \pm 0.001$, with the uncertainty due to the 1% inaccuracy of the tabulated values of $q(r)$ in Batchelor and Green (1972b). Finally, substituting these results into Eq. (39), we obtain:

$$\alpha = 1.20 \pm 0.005$$

Therefore, although the use of the exact probability distribution has almost doubled the numerical value of the drift velocity, it has not altered the qualitative feature of our main result, showing that the mean longitudinal drift velocity of the test sphere is proportional to the mean concentration gradient and is independent of the local concentration.

The case of general shear flow can be solved following the procedure described in the previous section, finding

$$\langle \mathbf{U} \rangle = \beta a^2 \mathbf{E} \cdot \nabla \phi \tag{42}$$

where $\beta = 2(\alpha_\infty + \alpha' - \alpha'')$. Now, however, unlike α_∞, which is a constant, α' and α'' will depend on the particular shear flow considered, since this determines the region of closed trajectories constituting the domain of integration in Eqs. (40) and (41). More accurately, for two-dimensional ambient flow fields we find that $\beta = \beta(\omega)$, where $\omega = |\det(\Omega)| / |\det(\Gamma)|$, with $0 \leqslant \omega \leqslant 1$, denoting the relative strength of the rotational component of the shear tensor. For example, for simple shear flow, i.e. $\omega = 1/2$, we found that $\beta(1/2) = 2.40 \pm 0.01$. In addition, for $\omega = 1$, the fluid undergoes pure rigid-body rotation and all particle trajectories are closed, so that we obtain $\alpha'' = \alpha_\infty$, $\alpha' = 0$ and $\beta(1) = 0$, as expected. Another interesting example is for $\omega = 0$, where the fluid undergoes pure straining flow. In this case, all the trajectories of one sphere relative to another originate from infinity and are open; hence $\alpha'' = 0$, while α' in Eq. (40) is evaluated with $g(\hat{r}) = 0$, with the result that, now, $\alpha' = 0.924$ and $\beta(0) = 3.12 \pm 0.01$.

4. Conclusions

We have presented the exact analytical calculation of the particle drift velocity $\langle \mathbf{U} \rangle$ for a dilute, neutrally buoyant suspension of spheres under the action of shear, due to the effect of an imposed concentration gradient. We found that $\langle \mathbf{U} \rangle$ is proportional to the concentration gradient $\nabla\phi$ through the relation $\langle \mathbf{U} \rangle = \beta a^2 \mathbf{E} \cdot \nabla \phi + O(\phi \nabla \phi)$, where a is the radius of the spheres and $\mathbf{E}$ is the fluid rate of strain, while β is an $O(1)$ constant that depends on the angular velocity of the fluid flow. In particular, $\beta = 2.40$ for simple shear flow, and $\beta = 3.12$ for pure straining flow. Now, the volumetric particle flux $\mathbf{J}$ at location $\mathbf{r}$ is related to the particle drift velocity through the following equality:

$$\mathbf{J} = (\langle \mathbf{U} \rangle + \mathbf{v}(\mathbf{r}))\phi - \mathbf{D}^{\mathrm{s}} \cdot \nabla \phi \tag{43}$$

where $\mathbf{v}(\mathbf{r})$ is the unperturbed fluid velocity, while $\mathbf{D}^{\mathrm{s}}$ is the self diffusivity tensor. This, in turn, is equal to the temporal growth rate of the second moment of the particle displacement, i.e.

$$\mathbf{D}^{\mathrm{s}} = \lim_{\Delta t \to \infty} \frac{\langle \Delta \mathbf{X} \Delta \mathbf{X} \rangle}{2 \Delta t} \tag{44}$$

with $\Delta \mathbf{X}$ denoting the net displacement of the test sphere in a macroscopically homogeneous suspension during a time interval Δt. In the case of simple shear flow $\mathbf{v}(\mathbf{r}) = \gamma x_2 \hat{\mathbf{e}}_1$, and when a concentration gradient is imposed along the $\hat{\mathbf{e}}_2$-direction, the longitudinal particle flux at the origin is $J_1 = \phi_0 \langle U_1 \rangle - D^{\mathrm{s}}_{12} \langle \partial\phi / \partial x_2 \rangle$. Therefore, since $\langle \Delta X_1 \Delta X_2 \rangle = O(\phi^2)$, we obtain, up to terms of $O(\phi^2)$ and $O(|a\nabla\phi|^2)$,

$$J_1 = -D_{12} \left\langle \frac{\partial \phi}{\partial x_2} \right\rangle \tag{45}$$

where

$$D_{12} = -1.20\gamma a^2 \phi_0 \tag{46}$$

is the particle shear-induced cross gradient diffusivity.

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